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EQUITABLE COLORING OF SPARSE PLANAR GRAPHS*

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Abstract. A proper vertex coloring of a graph G is equitable if the sizes of color classes differ by at most one. The equitable chromatic threshold $\chi^*_{eq}(G)$ of G is the smallest integer m such that G is equitably n-colorable for all $n \geq m$. We show that for planar graphs G with minimum degree at least two, $\chi^*_{eq}(G) \leq 4$ if the girth of G is at least 10, and $\chi^*_{eq}(G) \leq 3$ if the girth of G is at least 14.

Key words. equitable coloring, planar graphs, girth

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1. Introduction.

Graph coloring is a natural model for scheduling problems. Given a graph $G = (V, E)$, a proper vertex k-coloring is a mapping $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$ if $uv \in E(G)$. The notion of equitable coloring is a model to equally distribute resources in a scheduling problem. A proper k-coloring f is equitable if

$$|V_1| \leq |V_2| \leq \cdots \leq |V_k| \leq |V_1| + 1,$$

where $V_i = f^{-1}(i)$ for $i \in \{1, 2, \ldots, k\}$.

The *equitable chromatic number* $\chi_{eq}(G)$ of G is the smallest integer m such that G is equitably m-colorable. The *equitable chromatic threshold* of G, denoted by $\chi^*_{eq}(G)$, is the smallest integer m such that G is equitably n-colorable for all $n \geq m$. Note that $\chi_{eq}(G) \leq \chi^*_{eq}(G)$ for every graph G, and the two values may be different: for example, $\chi_{eq}(K_{7,7}) = 2$, while $\chi^*_{eq}(K_{7,7}) = 8$.

Hajnal and Szemerédi [2] proved that $\chi^*_{eq}(G) \leq \Delta(G) + 1$ for any graph G with maximum degree $\Delta(G)$. The following conjecture made by Chen, Lih, and Wu [1], if true, strengthens the above result.

CONJECTURE 1.1 (Chen, Lih, and Wu [1]). *For any connected graph G different from K_m, C_{2m+1} and $K_{2m+1,2m+1}$, $\chi^*_{eq}(G) \leq \Delta(G)$.*

Except for some special cases, the conjecture is still wide open in general.

Another direction of research on equitable coloring is to consider special families of graphs. For planar graphs, Zhang and Yap [5] proved that a planar graph is equitably m-colorable if $m \geq \Delta(G) \geq 13$. When the girth $g(G)$ is large, fewer colors are needed.

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THEOREM 1.1 (Wu and Wang [4]). *Let G be a planar graph with $\delta(G) \geq 2$.*
(a) *If $g(G) \geq 26$, then $\chi^*_{eq}(G) \leq 3$.*
(b) *If $g(G) \geq 14$, then $\chi^*_{eq}(G) \leq 4$.*

The purpose of this paper is to improve the above two results. Our main results are contained in the following theorems.

THEOREM 1.2. *If G is a planar graph with $\delta(G) \geq 2$ and $g(G) \geq 10$, then $\chi^*_{eq}(G) \leq 4$.*

THEOREM 1.3. *If G is a planar graph with $\delta(G) \geq 2$ and $g(G) \geq 14$, then $\chi^*_{eq}(G) \leq 3$.*

Since $K_{1,n}$ is not equitably k-colorable when $n \geq 2k-1$, we cannot drop the requirement of $\delta(G) \geq 2$ in the theorems. On the other hand, we do not believe that the girth conditions are best possible. Note that $\chi^*_{eq}(K_{2,n}) = \lceil \frac{n}{3} \rceil + 1$ for $n \geq 2$ and that the girth of $K_{2,n}$ is 4. It would be interesting to find the best possible girth condition for both 3- and 4-equitable colorings.

Last, let us note that, actually, we do not use planarity but only the weaker assumption that the graphs have maximum average degree less than 2.5 for Theorem 1.2 and less than 7/3 for Theorem 1.3. We do need the girth conditions, however, not only to control the density but also to ensure that the minimum degree is at least 2 at all times when we do reductions.

2. Preliminaries.

Before starting, we introduce some notation. In the whole paper, we take $1, 2, \ldots, m$ to be the set of integers modulo m. A *k-vertex* is a vertex of degree k; a k^+- and a *k^--vertex* have degree at least and at most k, respectively. A *thread* is either (a) a path with 2-vertices in its interior and 3^+-vertices as its endvertices or (b) a cycle with exactly one 3^+-vertex and all other vertices of degree 2 (in other words, case (a) with endvertices equal). A *k-thread* has k interior 2-vertices. If a 3^+-vertex u is the endvertex of a thread containing a 2-vertex v and the distance between u and v on the thread is $l+1$, then we say that u and v are *loosely l-adjacent*. Thus "loosely 0-adjacent" is the same as the usual "adjacent."

All of our proofs rely on the techniques of reducibility and discharging. We start with a minimal counterexample G to the theorem we are proving, and the idea of the reduction is as follows. We remove a small subgraph H (for instance, a vertex of degree at least three, together with its incident 2-threads) from the graph G. By the minimality of G, we therefore have an equitable k-coloring f of $G-H$, and we attempt to extend f to an equitable coloring of G. This can be done if we can equitably k-color H itself with some extra conditions, namely, the color classes which should be "large" in H are predetermined by the existing coloring of $G-H$, and second, the parts of H with edges to $G-H$ have color restrictions. If every equitable k-coloring of $G-H$ can be extended into an equitable k-coloring of G, then H is called a *reducible configuration*.

We will handle the latter condition by means of lists of allowed colors in H. We will handle the former condition by predetermining the sizes of the color classes. Thus we have the following definition.

DEFINITION 2.1. *Let H be a graph with list assignment $L = \{l_v\}$ with $l_v \subseteq \{1, 2, \ldots, m\}$. The graph H is* descending-equitably L-colorable *if H can be L-colored such that $|V_1| \geq |V_2| \geq \cdots \geq |V_m| \geq |V_1| - 1$.*

Note that if $G-H$ has an equitable k-coloring with $|V_1| \leq |V_2| \leq \cdots \leq |V_k| \leq |V_1| + 1$, then G is equitably k-colorable if H is descending-equitably L-colorable. Because of this, a descending-equitably L-colorable subgraph H is a reducible configuration in G. Regarding the lists ℓ_v, we always take ℓ_v to be the set of all colors not assigned to any neighbor of v in $G-H$.

The *maximum average degree* of G is $mad(G) = \max\{ \frac{2|E(H)|}{|V(H)|} \mid H \subseteq G\}$. A planar graph G with girth at least g has maximum average degree less than $\frac{2g}{g-2}$. We let the initial charge at vertex v be $M(v) = d(v) - \frac{2g}{g-2}$. We will introduce some rules to redistribute the charges (discharging), and after the discharging process, every vertex v has a final charge $M'(v)$. Note that

$$\sum_{v \in V(G)} M'(v) = \sum_{v \in V(G)} M(v) = \sum_{v \in V(G)} \left(d(v) - \frac{2g}{g-2} \right) < 0. \tag{1}$$

We will show that either we have some reducible configurations or the final charges are all nonnegative. The former contradicts the assumption that G is a counterexample, and the latter contradicts (1).

We will prove the theorems on 3-coloring and 4-coloring separately. Before the proofs, we provide some properties useful to equitable m-coloring with $m \geq 3$.

Let $m \geq 3$ be an integer. Let G be a graph that is not equitably m-colorable with $|V| + |E|$ as small as possible.

Observation 2.1. The graph G is connected.

Proof. Let $H_1, H_2, \ldots, H_k$ be the connected components of G, where $k \geq 2$. By the minimality of G, both $H = H_1 \cup H_2 \cup \cdots \cup H_{k-1}$ and H_k are equitably m-colorable. An equitable m-coloring of H with $|V_1(H)| \geq |V_2(H)| \geq \cdots \geq |V_m(H)|$ and an equitable m-coloring of H_k with $|V_1(H_k)| \leq |V_2(H_k)| \leq \cdots \leq |V_m(H_k)|$ induce an equitable m-coloring of G, which contradicts the choice of G. ∎

3. Equitable 4-coloring. In this section, we prove Theorem 1.2. We start with some useful lemmas. The following lemma is an extension of a fact first observed by Kostochka, Pelsmajor, and West [3].

LEMMA 3.1. *Fix a positive integer m, and let $t \in \{1, 2, \ldots, m\}$. Let $S = \{x_1, x_2, \ldots, x_t\}$ be a set of t distinct vertices in G with $t \leq m$. Let $L = \{\ell_v\}_{v \in S}$ be a list assignment with $\ell_v \subseteq \{1, 2, \ldots, m\}$ for all $v \in S$. If $G - S$ has an equitable m-coloring f and*

$$|\ell_{x_i}| \geq |N_G(x_i) \setminus S| + i \tag{2}$$

for $i \in \{1, 2, \ldots, t\}$, then f can be extended into an equitable m-coloring of G.

Proof. Let $G_i = G - \{x_{i+1}, \ldots, x_t\}$ for $i \in \{0, 1, \ldots, t-1\}$. Then $G_0 = G - S$, and $G_m = G$. Starting from an equitable coloring of G_0, we extend it to $G_1, G_2, \ldots, G_t$ in this order. Suppose that we are to color x_{i+1}, given an equitable coloring of G_i. By (2), we can give a color to x_{i+1} that is in $l_{x_{i+1}}$ and is different from the colors used on $x_1, x_2, \ldots, x_i$. By construction, the colors used on S are all different; hence the coloring of $G_t = G$ is an equitable m-coloring. ∎

LEMMA 3.2. *Let G be a graph and $P = y_0 y_1 \ldots y_t y_{t+1}$ such that $t \in \{4, 5\}$ and $d(y_i) = 2$ for each $i \in \{1, 2, \ldots, t\}$. Let $m \geq 4$ be an integer and $a, b \in \{1, 2, \ldots, m\}$. Let x be an arbitrary vertex in $\{y_1, y_2, \ldots, y_t\}$. If $G - \{y_1, \ldots, y_t\}$ has an equitable m-coloring f, then f can be extended to an equitable m-coloring of G such that $f(x) \notin \{a, b\}$ unless $m = 4$, $t = 5$, and $x \in \{y_2, y_4\}$.*

Proof. Let $V_1, \ldots, V_m$ be the m color classes of $G - S$ under f with $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$, where $S = \{y_1, y_2, \ldots, y_t\}$. By symmetry, we may assume that $x = y_i$ with $i \leq \lceil \frac{t}{2} \rceil$.

When $m \geq t$, we arrange the vertices y_j into a list $x_1, x_2, \ldots, x_t$ such that $x_1 = x$ and $x_t \notin \{y_1, y_t\}$, and we assign every vertex other than x with the same color list

$\{1, 2, \ldots, t\}$. Let $l_x = \{1, 2, \ldots, t\} \setminus \{a, b\}$. Then, by Lemma 3.1, we can extend f to G such that $f(x) \notin \{a, b\}$.

If $m < t$, then $m = 4$ and $t = 5$, and in this case, $x \in \{y_1, y_3\}$. If $1 \notin \{a, b, f(y_0)\}$, then assign 1 to y_1 and y_3, assign a color $c \in \{2, 3, 4\} \setminus \{f(y_6)\}$ to y_5, and assign the other two colors in $\{2, 3, 4\} \setminus \{c\}$ arbitrarily to y_2 and y_4. If $1 \in \{f(y_0), a, b\}$, then $|\{2, 3, 4\} - \{f(y_0), a, b\}| \geq 1$. Let $x' \in \{y_1, y_3\} \setminus \{x\}$ and $c_2 \in \{2, 3, 4\} \setminus \{f(y_0), a, b\}$. Assign 1 to y_2 and y_4, assign c_2 to x, assign a color $c_3 \in \{2, 3, 4\} \setminus \{c_2, f(y_0)\}$ to x', and assign the remaining color $c_4 \in \{2, 3, 4\} \setminus \{c_2, c_3\}$ to y_5. If $c_4 = f(y_6)$, then swap colors on y_5 and y_4, i.e., recolor y_4 with c_4 and y_5 with 1. In either case, f is extended to G, a contradiction. □

LEMMA 3.3. *Let xy_1y_2y be a 2-thread of G and $m \geq 4$ be an integer. If $G - \{y_1, y_2\}$ has an m-equitable coloring f such that $f(x) \neq f(y)$, then f can be extended to an equitable m-coloring of G.*

Proof. Let f be an equitable m-coloring of $G - \{y_1, y_2\}$, and let $V_1, \ldots, V_m$ be the m color classes with $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$. If $f(x) \neq f(y)$, then there is a bijection $\phi : \{1, 2\} \to \{1, 2\}$ such that $\phi(1) \neq f(x)$ and $\phi(2) \neq f(y)$. Assign $\phi(1)$ to y_1 and $\phi(2)$ to y_2. Hence f can be extended to G. □

LEMMA 3.4. *Let xy_1y_2y be a 2-thread and xy_3z be a 1-thread incident with x. Let $m \geq 4$ be an integer. If $G - \{y_1, y_2, y_3\}$ has an equitable m-coloring f with $f(x) \notin \{f(y), f(z)\}$, then f can be extended to an equitable m-coloring of G.*

Proof. Let $V_1, V_2, \ldots, V_m$ be the m color classes with $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$. If $f(x) \in \{1, 2, 3\}$, then let $a = f(x)$, so $a \neq f(y)$. Otherwise, let $a \in \{1, 2, 3\} \setminus \{f(y)\}$. Let $b \in \{1, 2, 3\} \setminus \{a, f(z)\}$ and $c \in \{1, 2, 3\} \setminus \{a, b\}$. Then $b \notin \{f(x), f(z)\}$, $c \neq f(x)$, and $\{a, b, c\} = \{1, 2, 3\}$. Assigning a to y_2, b to y_3, and c to y_1 yields an equitable m-coloring of G, a contradiction. □

Proof of Theorem 1.2. Let G be a minimal counterexample to Theorem 1.2 with $|V| + |E|$ as small as possible. That is, G is a planar graph with $\delta(G) \geq 2$ and girth at least 10, and G is not equitably m-colorable for some integer $m \geq 4$, but every proper subgraph of G with minimum degree at least 2 is equitably m-colorable for each $m \geq 4$. □

CLAIM 3.1. *The graph G has no t-thread with $t \geq 3$, and G has no thread whose endvertices are identical.*

Proof. Suppose, on the contrary, that G has a t-tread $P = v_0v_1 \ldots v_tv_{t+1}$ with $t \geq 3$, where $d(v_0), d(v_{t+1}) \geq 3$.

If $v_0 \neq v_{t+1}$ or $d(v_0) \geq 4$, consider $G_1 = G - \{v_1, \ldots, v_t\}$. Then $\delta(G_1) \geq 2$. By the minimality of G, the graph G_1 has an equitable m-coloring. Let $V_1, V_2, \ldots, V_m$ be the m color classes with $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$. We can extend the coloring to G to obtain an equitable m-coloring of G as follows: First color the vertex v_i by the color k, where $k \equiv i \pmod{m}$ for each $i \in \{1, 2, \ldots, t\}$. Swap the colors of v_1 and v_2 if the colors of v_1 and v_0 are the same, and further swap the colors of v_{t-1} and v_t if the colors of v_t and v_{t+1} are the same.

Now assume that $v_0 = v_{t+1}$ and $d(v_0) = 3$. Let $x \in N(v_0) \setminus \{v_1, v_t\}$. If $d(x) \geq 3$, consider $G_2 = G - \{v_0, v_1, \ldots, v_t\}$. Then $\delta(G_2) \geq 2$. By the choice of G, the graph G_2 has an equitable m-coloring with color classes $V_1, V_2, \ldots, V_m$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$. Since $q \geq 1$, we can extend the coloring to G to obtain an equitable m-coloring of G as follows: First color the vertex v_i by the color k, where $k \equiv i \pmod{m}$. If $0 \equiv t \pmod{m}$, swap the colors of v_t and v_{t-1}. If the colors of x and v_0 are the same, further swap the colors of v_0 and v_i, where $i \in \{1, 2\}$ such that the color of v_i is different from that of v_t (such a vertex v_i exists since v_0, v_1, and v_2 are colored differently).

If $d(x) = 2$, then let $Q = x_0x_1 \dots x_qx_{q+1}$ be the thread containing the edge v_0x_1, where $x_1 = x$ and $x_0 = v_0$. Consider the graph $G_3 = G - \{v_0, x_1, \dots, x_q, v_1, \dots, v_t\}$. Then $\delta(G_3) \geq 2$. By the minimality of G, the graph G_3 has an equitable m-coloring with color classes $V_1, V_2, \dots, V_m$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$. We first extend the coloring to G_2 to obtain an equitable m-coloring of $G - \{v_1, \dots, v_t\}$ as follows: First color the vertex x_i by the color k, where $k \equiv i + 1 \pmod m$ for each $i \in \{0, 1, \dots q\}$. If x_q and x_{q+1} have the same color, swap the colors of x_q and x_{q-1}. Next we further extend the coloring to G similarly to the case that $d(v_0) \geq 4$. ∎

Let x be a vertex of degree $d = d(x) \geq 3$. Then x is the endvertex of d threads. Set $T(x) = (a_2, a_1, a_0)$, where a_i is the number of i-threads incident with x. Let $t(x) = 2a_2 + a_1$. Claim 3.1 implies that $t(x)$ is the number of 2-vertices loosely adjacent to x.

CLAIM 3.2. *If x is a 4-vertex, then $t(x) \leq 5$.*

Proof. Suppose, on the contrary, that x is a 4-vertex with $t(x) \geq 6$. Claim 3.1 implies that x is not incident with any t-thread such that $t \geq 3$. Since $t(x) \geq 6$, the vertex x is incident with at least two 2-threads. Label two 2-threads incident with x as $xx_1z_1y_1$ and $xx_2z_2y_2$.

We first show that x is incident with at most two 2-threads. Suppose that x is incident with a third 2-thread $xx_3z_3y_3$. Label the fourth thread incident with x as $xx_4z_4y_4$, xz_4y_4, or xy_4, depending on whether it is a 2-thread, a 1-thread, or a 0-thread. Set $A = \{x, x_i, z_i \mid 1 \leq i \leq 4\}$, $A = \{x, x_j, z_i \mid 1 \leq i \leq 4, 1 \leq j \leq 3\}$, or $A = \{x, x_i, z_i \mid 1 \leq i \leq 3\}$, depending on whether x is incident with four 2-threads, a 1-thread, or a 0-thread, respectively. Since $g(G) \geq 10$, the threads do not share endvertices other than x, so $\delta(G - A) \geq 2$. By the minimality of G, the graph $G - A$ has an equitable m-coloring f.

Now if x is not incident with a 0-thread, then by Lemma 3.2, f can be extended to $G - \{x_1, x_2, z_1, z_2\}$ such that $f(x) \notin \{f(y_1), f(y_2)\}$. By Lemma 3.3, it can be further extended to $G - \{x_1, z_1\}$ since $f(x) \neq f(y_2)$ and to G since $f(x) \neq f(y_1)$. This contradicts the choice of G.

If, on the other hand, x is incident with a 0-thread, then first extend the coloring f of $G - A$ to $G - \{x_1, z_1\} - xy_4$ such that $f(x) \notin \{f(y_1), f(y_4)\}$ by Lemma 3.2. Since $f(x) \neq f(y_4)$, it is also an equitable m-coloring of $G - \{x_1, z_1\}$. Since $f(x) \neq f(y_1)$, by Lemma 3.3, the coloring of $G - \{x_1, z_1\}$ can be extended to G, a contradiction. This proves that x is incident with at most two 2-threads.

Therefore, $t(x) \leq 6$, and hence $t(x) = 6$. Thus, $T(x) = (2, 2, 0)$. Label the two 1-threads incident with x as xx_3y_3 and xx_4y_4. Then $G - \{x, z_1, z_2, x_i \mid 1 \leq i \leq 4\}$ has an equitable m-coloring. Since $y_3x_3xx_2z_2y_2$ is a 4-thread in $G - \{x_1, z_1, x_4\}$, Lemma 3.2 implies that f can be extended to $G - \{x_1, z_1, x_4\}$ such that $f(x) \notin \{f(y_1), f(y_4)\}$. By Lemma 3.4, it can be further extended to G. This contradicts the choice of G, thereby proving Claim 3.2. ∎

CLAIM 3.3. *For a 3-vertex x, either $t(x) \leq 2$ or $T(x) = (1, 2, 0)$ and $m = 4$.*

Proof. We first prove that $T(x) \neq (1, 2, 0)$ if $m \geq 5$. Suppose, on the contrary, that $T(x) = (1, 2, 0)$ and $m \geq 5$. Label the two 1-threads incident with x as xx_1y_1 and xx_2y_2, and label the 2-thread as $xx_3x_4y_3$. Let $A = \{x, x_1, x_2, x_3, x_4\}$. Then $\delta(G - A) \geq 2$, and it has an equitable m-coloring f with color classes $V_1, V_2, \dots, V_m$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$. Let $\{a, b, c, d, e\} = \{1, 2, 3, 4, 5\}$ such that $a \neq f(y_1)$, $b \neq f(y_2)$, and $c \neq f(y_3)$. Assigning a to x_1, b to x_2, c to x_4, d to x, and e to x_3 yields an equitable m-coloring of G, a contradiction.

Now we prove that if $T(x) \neq (1, 2, 0)$, then $t(x) \leq 2$. Suppose, on the contrary, that $t(x) \geq 3$ and $T(x) \neq (1, 2, 0)$. Claim 3.1 implies that x is not incident with any

t-thread where $t \geq 3$. We first consider the case where x is not incident with a 2-thread. Then $T(x) = (0, 3, 0)$. Label the three 1-threads incident with x as xx_iy_i, where $d(x_i) = 2$ and $d(y_i) \geq 3$ for $i \in \{1, 2, 3\}$. Consider the graph $G_1 = G - \{x, x_1, x_2, x_3\}$. Then $\delta(G_1) \geq 2$, so by the minimality of G, the graph G_1 has an equitable m-coloring with color classes $V_1, V_2, \ldots, V_m$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$. Let $\{1, 2, 3, 4\} = \{a, b, c, d\}$ such that no color in $\{a, b, c\}$ is used by all three vertices y_1, y_2, y_3. An equitable m-coloring of G can be obtained by coloring the vertices x_1, x_2, x_3 with the colors a, b, c such that no conflict occurs and coloring the vertex x with the color d. This contradiction shows that $T(x) \neq (0, 3, 0)$, and hence x is incident with at least one 2-thread.

Now we consider the case that $a_2 \neq 0$. Let xx_1x_2y be a 2-thread incident with x. If $t(x) \geq 5$, then $G - \{x_1, x_2\}$ has minimum degree 2 and has a t-thread P that contains x for some $t \in \{4, 5\}$. Let G_2 be the subgraph obtained from $G - \{x_1, x_2\}$ by further deleting the degree-2 vertices in P. Then G_2 has an equitable m-coloring f. By Lemma 3.2, f can be extended to $G - \{x_1, x_2\}$ such that $f(x) \neq f(y)$. By Claim 3.3, f can be further extended to G. This contradiction shows that $3 \leq t(x) \leq 4$. Since x is incident with at least one 2-thread and $T(x) \neq (1, 2, 0)$, the vertex x must be incident with a 0-thread. Call it xu. Since $t(x) \geq 3$, the graph $G - xu$ has a t-thread P that contains x with $t \in \{4, 5\}$. Let G_3 be the subgraph obtained from $G - xu$ by further deleting the degree-2 vertices in P. Then G_3 has an equitable m-coloring f. Lemma 3.2 implies that f can be extended to $G - xu$ such that $f(x) \neq f(u)$. This extension of f is also an equitable m-coloring of G, a contradiction. This completes the proof of Claim 3.3. ∎

A 3-vertex x in G is *bad* if $T(x) = (1, 2, 0)$. Note that if $m \geq 5$, the configuration $T(x) = (1, 2, 0)$ with $d(x) = 3$ is still reducible; thus there are no bad 3-vertices when $m \geq 5$. The following claim deals with reducible configurations for $m = 4$.

CLAIM 3.4. *Assume $m = 4$. Let x be a bad 3-vertex and y be a vertex that is loosely 1-adjacent to x. Then*
(1) *if $d(y) = 3$, then $t(y) = 1$;*
(2) *if $d(y) = 4$, then y is loosely 1-adjacent to exactly one bad 3-vertex, namely, x.*

Proof. Label the threads incident with x as $xx_1x_2u_1$, xx_3u_2, and xx_4y. Here and after in the proof, we always assume that the vertices x_i, y_i, and z_i have degree 2, while the vertices u_i have degree at least 3.

(1) Suppose that $d(y) = 3$ and $t(y) \geq 2$. Then Claim 3.3 ensures that either $t(y) = 2$ or y is a bad 3-vertex. If $t(y) = 2$, then $T(y) = (0, 2, 0)$, while if y is a bad 3-vertex, then $T(y) = (1, 2, 0)$. In either case, y is incident with exactly two 1-threads. Label the other 1-thread incident with y as yy_1z. Label the third thread incident with y as yu_0 or $yy_2y_3u_0$, depending on whether it is a 0-thread or a 2-thread. Set $A = \{x, y, y_1, x_i \mid 1 \leq i \leq 4\}$. Let $B = \emptyset$ if y is incident with a 0-thread and $B = \{y_2, y_3\}$ otherwise. Consider the graph $G_1 = G - (A \cup B)$. Since $g(G) \geq 10$, the vertices u_1, u_2, z, u_0 are distinct, and thus $\delta(G_1) \geq 2$. By the minimality of G, the graph G_1 has an equitable 4-coloring. Note that any 4-equitable coloring of G_1 can be extended to $G - A$, and let f be an equitable 4-coloring of $G - A$. We color x, y, y_1, and x_4 in this order as follows: Pick one color c_1 for x in $\{1, 2, 3, 4\} \setminus \{f(u_1), f(u_2)\}$, c_2 for y in $\{1, 2, 3, 4\} \setminus \{c_1, c(y_2)\}$ if $B \neq \emptyset$ and in $\{1, 2, 3, 4\} \setminus \{c_1, c(u)\}$ if $B = \emptyset$, c_3 for y_1 in $\{1, 2, 3, 4\} \setminus \{c_1, c_2, c(z)\}$, and c_4 for x_4 in $\{1, 2, 3, 4\} \setminus \{c_1, c_2, c_3\}$. In such a way, f can be extended to an equitable coloring of $G - \{x_1, x_2, x_3\}$ such that $f(x) \notin \{f(u_1), f(u_2)\}$.

By Lemma 3.4, the equitable 4-coloring of $G-\{x_1, x_2, x_3\}$ can be further extended to G, which contradicts the choice of G, and hence we prove (1).

(2) Suppose that $d(y) = 4$ and that y is loosely 1-adjacent to two bad 3-vertices x and z. Label the threads incident with z as $zz_1z_2u_3$, zz_3u_4, and zy_1y. Let u_5 and u_6 be the endvertices of the two threads incident with y other than the ones incident to x and z. Set $A = \{x, y, z, y_1, x_i, z_j \mid 1 \leq i \leq 4, 1 \leq j \leq 3\}$. Let B be the set of 2-vertices on the two threads incident with y other than yy_1z and yx_4x. Let $G_1 = G - (A \cup B)$. Observe that, since $g(G) \geq 10$, all the named vertices are distinct except, possibly, u_1 and u_3. Consequently, either $\delta(G_1) \geq 2$ or $u_1 = u_3$, and u_1 has degree 3 in G. However, this last case contradicts Claim 3.3. Thus, $\delta(G_1) \geq 2$, and hence G_1 has an equitable 4-coloring f with color classes V_1, V_2, V_3, V_4 such that $|V_1| \leq |V_2| \leq |V_3| \leq |V_4|$. Note that if y is incident with a 2-thread, then f can be extended to the 2-vertices in the 2-thread. This is why, in the following, we may assume, without loss of generality, that y is not incident with a 2-thread.

We first consider the case where y is incident with exactly two 1-threads: xx_4y and zy_1y. Then $B = \emptyset$. Using Lemma 3.2, we extend f to y_1yx_4x such that $f(y) \notin \{f(u_5), f(u_6)\}$. Note that the colors $f(x)$, $f(x_4)$, $f(y)$, and $f(y_1)$ are distinct. If $f(x) \in \{f(u_1), f(u_2)\}$, then one of $f(x_4)$ and $f(y_1)$ is not in $\{f(u_1), f(u_2)\}$. If $f(x_4) \notin \{f(u_1), f(u_2)\}$, then swap the colors of $f(x)$ and $f(x_4)$. If $f(y_1) \notin \{f(u_1), f(u_2)\}$, then swap the colors of $f(x)$ and $f(y_1)$. Hence we have an extension of f on xx_4yy_1 such that $f(x) \notin \{f(u_1), f(u_2)\}$. By Lemmas 3.2 and 3.4, f can be further extended to G, a contradiction.

Now we consider the case where y is incident with at least three 1-threads. Label the third 1-thread as yy_2u_5 and the fourth thread incident with y as yy_3u_6 or yu_6, depending on whether it is a 1-thread or a 0-thread. Note that either $B = \{y_2\}$ or $B = \{y_2, y_3\}$. We first extend f to $\{x_4, y, y_1\} \cup B$. Let a and b be two distinct colors in $\{1, 2, 3, 4\} \setminus \{f(u_5), f(u_6)\}$. Assign a to y_2. If $B = \{y_2\}$, then assign b to y_1; otherwise, assign b to y_3. Now assign each of the colors of $\{1, 2, 3, 4\} \setminus \{a, b\}$ arbitrarily, making sure that both x and y_1 are colored 1 if $B = \{y_2, y_3\}$ and $1 \notin \{a, b\}$. This yields an equitable 4-coloring of G, a contradiction. ∎

Since $g(G) \geq 10$, we have $mad(G) < 2.5$. Let $M(x) = d(x) - 2.5$ be the *initial charge* of x for $x \in V$. We will redistribute the charges among vertices according to the *discharging rules* below.

(R1) Each 2-vertex receives $\frac{1}{4}$ from each of the endvertices of the thread containing it.

(R2) Each bad 3-vertex receives $\frac{1}{4}$ from each of the vertices that are loosely 1-adjacent to it.

Let $M'(x)$ be the charge of x after application of rules R1 and R2. The following claim shows a contradiction to (1), which implies the truth of Theorem 1.2.

CLAIM 3.5. *$M'(x) \geq 0$ for each $x \in V$.*

Proof. Let $x \in V$. If $d(x) = 2$, then $M'(x) = 2 - 2.5 + \frac{2}{4} = 0$.

Assume $d(x) = 3$ and that x is not a bad vertex. If x is not loosely 1-adjacent to any bad vertex, then Claim 3.3 ensures that $t(x) \leq 2$, so x sends out at most $2 \times \frac{1}{4} = \frac{1}{2}$. If x is loosely 1-adjacent to a bad vertex, then Claim 3.4 implies that $t(x) = 1$, so x sends out at most $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. In either case, $M'(x) \geq 3 - 2.5 - \frac{1}{2} = 0$.

Assume $d(x) = 3$ and that x is a bad vertex. Then $t(x) = 4$, and x sends out $4 \times \frac{1}{4} = 1$. It also receives $\frac{1}{4}$ from each loosely 1-adjacent vertex, of which there are 2. Hence $M'(x) \geq 3 - 2.5 - 1 + 2 \times \frac{1}{4} = 0$.

Assume $d(x) = 4$. Then x is loosely 1-adjacent to at most one bad 3-vertex by Claim 3.4. Hence x sends out at most $\frac{t(x)+1}{4} \leq \frac{3}{2}$ since $t(x) \leq 5$ by Claim 3.2. Therefore $M'(x) \geq 4 - 2.5 - \frac{3}{2} = 0$.

Assume $d(x) \geq 5$. Let y be a 3^+-vertex that is loosely k-adjacent to x. If $k = 2$, then x sends out $2 \times \frac{1}{4}$ via this 2-thread. If $k = 1$, then x sends out $\frac{1}{4}$ via this thread if y is not a bad vertex and sends out $2 \times \frac{1}{4} = \frac{1}{2}$ via this 1-thread if y is a bad vertex. In summary, x sends out at most $\frac{1}{2}$ via each thread incident with it. Hence $M'(x) \geq d(x) - 2.5 - \frac{d(x)}{2} = \frac{d(x)}{2} - 2.5 \geq 0$. ☐

4. Reduction lemmas for equitable 3-coloring. We now proceed to equitable 3-coloring. We first prove two lemmas which give conditions for the existence of reducible configurations.

A subdivided star H is a graph obtained from a star by replacing the edges by paths. (We will call these paths "threads" as well.) In our reducible configurations, we will see the natural connections: If we take a vertex v with the 2-vertices on its incident threads in graph G, we obtain a subdivided star with root v. So in the following two lemmas, even though we state and prove them as graphs, they are, indeed, part of the graphs under consideration.

Let a_i^v be the number of i-threads incident to vertex v. If it is clear from the context, we drop v in the notation. The two lemmas that follow give simple ways to identify reducible configurations using relations involving a_i^v.

Remark 4.1. In the following lemma, the fact that we assume only two allowed colors at the root instead of three corresponds to the fact that we are allowing for one 3^+-vertex adjacent to the root (i.e., one 0-thread incident with the root).

LEMMA 4.1 (reducing a vertex with at most one 0-thread). *Let S be a subdivided star of order s with root x. Let $L = \{\ell_v\}$ be a list assignment to the vertices of S such that $\ell_v = \{1, 2, 3\}$ if v is neither a leaf nor the root, $\ell_v \subset \{1, 2, 3\}$ with $|\ell_v| = 2$ if v is a leaf, and $|\ell_x| \geq 2$. Let $d(x) \leq 6$, and assume $a_i = 0$ unless $i \in \{0, 1, 2, 4\}$. If $2a_4 + a_2 \geq a_1 + 1 + \varepsilon$ and $a_4 \geq d(x) - 4$, then S is descending-equitably L-colorable, where $\varepsilon = 3\lceil s/3\rceil - s$.*

Proof. Let c and c' be two colors allowed at x. Let p_i ($i \in \{1, 2, 3\}$) be the desired size of V_i. Let S_i ($i \in \{c, c'\}$) be a maximum independent set that contains the root and is such that $i \in l_v$ for all $v \in S_i$. Then no vertex of a 1-thread is in any of the S_i's; each 2-thread contains a leaf that is in at least one of S_i's; and for each 4-thread, the leaf is in at least one of the S_i's, and the vertex at distance 2 from the root is in both of the S_i's. Thus $|S_c| + |S_{c'}| \geq 2 + 3a_4 + a_2$.

We wish first to find a color for the roots that may be extended to an independent set of size $\lceil s/3\rceil$; the candidates for such a set are S_c and $S_{c'}$. Assume, for a contradiction, that they are both of size at most $\lceil s/3\rceil - 1$. Then, $2 + 3a_4 + a_2 \leq |S_c| + |S_{c'}| \leq 2(\lceil s/3\rceil - 1) = \frac{2}{3}(s + \varepsilon - 3) = \frac{2}{3}(4a_4 + 2a_2 + a_1 + 1 + \varepsilon - 3)$. Therefore, $a_4 + 12 \leq a_2 + 2(a_1 + \varepsilon + 1) \leq a_2 + 2(2a_4 + a_2)$, that is, $a_4 + a_2 \geq 4$. So $a_2 + 2(a_1 + \varepsilon + 1) \geq a_4 + 12 \geq 16 - a_2$, and hence $a_1 + a_2 + \varepsilon \geq 7$. Adding a_4 to both sides and observing that $\varepsilon \leq 2$ yields that $d(x) \geq a_1 + a_2 + a_4 \geq 5 + a_4$, whence $a_4 \leq d(x) - 5$, which contradicts our hypothesis that $a_4 \geq d(x) - 4$.

Thus there exists an independent set of size at least $\lceil s/3\rceil$ containing the root and having a common color available. Let c be that color, and fix a subset T_c of S_c of size exactly p_c, with the additional property that T_c contains all vertices of 4-threads that are at distance exactly 2 from the root. (This is possible because $a_4 < s/3$.)

Let c' and c'' be the other two colors. Without loss of generality, we may assume that $p_{c'} \geq p_{c''}$. We color with c' first. By construction, T_c contains no vertices at distance 1 or 3 from the root. There are $2a_4 + a_2$ such vertices that are not leaves. Assuming that $p_{c'} \geq \lceil s/3 \rceil$, we compare these quantities and find that $\lceil s/3 \rceil = \frac{4a_4+2a_2+a_1+1+\varepsilon}{3} \leq \frac{4a_4+2a_2+2a_4+a_2}{3} \leq 2a_4 + a_2$; thus, at worst, $2a_4 + a_2$ is exactly "big enough," and we assign a preliminary set $W_{c'}$ the color c' such that (a) all vertices at distance 3 from the root are in $W_{c'}$ and (b) some nonleaf vertices adjacent to the root are in $W_{c'}$ such that $W_{c'}$ has size $p_{c'}$. (This is possible because again $a_4 < s/3$.)

The remaining vertices, which we shall group together in a set $W_{c''}$, are "assigned" the color c'', with the caveat that these $W_{c''}$ vertices contain the leaves and thus may not have c'' in their list.

To pass, therefore, to a legitimate L-coloring, we pair the vertices of $W_{c''}$ that are leaves with a subset of $W_{c'}$ as follows. For each z that is a leaf of a 2-thread or a 4-thread, define z^* to be the neighbor of z. For each z that is a leaf of a 1-thread, we may assign a unique z^* such that z^* is a neighbor of the root and z^* lies in a 4-thread. (Note that this is possible because, from the second paragraph of this proof, $a_2 + a_4 \geq 4$, whence $a_4 \geq d(x) - 4 \geq a_4 + a_2 + a_1 - 4 \geq 4 + a_1 - 4 = a_1$.) Now in every case, $z \in W_{c''}$ and $z^* \in W_{c'}$ by construction; swap the colors on z and z^* if c'' is not in ℓ_z. Note that the obtained coloring is legitimate because, in each case, the other vertices adjacent to z^* received the color c. □

LEMMA 4.2 (reducing two vertices connected by a 1-thread; one vertex may have one 0-thread). *Suppose x and y are connected by a 1-thread and that $d(x)+d(y) \leq 8$. Let S be a graph of order s induced by the union of the subdivided star with root x and the subdivided star with root y. Let $L = \{\ell_v\}$ be a list assignment to the vertices of S such that $\ell_v = \{1, 2, 3\}$ if v is neither a leaf nor y, $\ell_v \subset \{1, 2, 3\}$ with $|\ell_v| = 2$ if v is a leaf, and $\ell_y \subseteq \{1, 2, 3\}$ with $|\ell_y| \geq 2$. Let $b_i = a_i^x + a_i^y$ for $i \in \{2, 4\}$, and let $b_1 = a_1^x + a_1^y - 1$. Then S is descending-equitably L-colorable if $2b_4 + b_2 \geq b_1 - 1 + \varepsilon$ and $b_4 \geq 1$, where $\varepsilon = 3\lceil s/3 \rceil - s$.*

Proof. Let c and c' be two colors allowed at y. Let p_i ($i \in \{1, 2, 3\}$) be the desired size of V_i. Let S_i ($i \in \{c, c'\}$) be a maximum independent set that contains x and y and is such that $i \in \ell_v$ for all $v \in S_i$. Then no vertex of a 1-thread is in any of the S_i's; each 2-thread contains a leaf that is in at least one of the S_i's; and for each 4-thread incident with x (y, respectively), the leaf is in at least one of the S_i's, and the vertex at distance 2 from x (y) is in both of the S_i's. Thus $|S_c| + |S_{c'}| \geq 4 + 3b_4 + b_2$.

We wish first to find a color for the root that may be extended to an independent set of size $\lceil s/3 \rceil$; the candidates for such a set are S_c and $S_{c'}$. Assume, for a contradiction, that they are both of size at most $\lceil s/3 \rceil - 1$. Then $4+3b_4+b_2 \leq |S_c|+|S_{c'}| \leq 2(\lceil s/3 \rceil - 1) = \frac{2}{3}(s+\epsilon-3) = \frac{2}{3}(4b_4+2b_2+b_1+2)$. Therefore, $b_4 + 14 \leq b_2 + 2(b_1 + \epsilon) \leq b_2 + 2(2b_4 + b_2) + 2$, that is, $b_4 + b_2 \geq 4$. So $b_2 + 2(b_1 + \epsilon) \geq b_4 + 14 \geq 18 - b_2$, and hence $b_1 + b_2 + \epsilon \geq 9$. Adding b_4 to both sides and observing that $\varepsilon \leq 2$ yields that $d(x)+d(y)+1 = b_1+b_2+b_4+2 \geq b_1+b_2+b_4+\varepsilon \geq 9+b_4 \geq 10$, whence $d(x) + d(y) \geq 9$, which contradicts our assumption that $d(x) + d(y) \leq 8$.

Thus there exists an independent set of size at least $\lceil s/3 \rceil$ containing x and y and having a common color available. Let c be that color, and fix a subset T_c of S_c of size exactly p_c, with the additional property that T_c contains all vertices of 4-threads incident with x or y that are at distance exactly 2 from x or y, respectively. (This is possible because $b_4 < s/3$.)

Let c' and c'' be the other two colors. Without loss of generality, we may assume that $p_{c'} \geq p_{c''}$. We color with c' first. By construction, T_c contains no vertices at distance 1 or 3 from either x or y. There are $2b_4 + b_2 + 1$ such vertices that are not leaves. Assuming that $p_{c'} \geq \lceil s/3 \rceil$, we compare these quantities and find that $\lceil s/3 \rceil = \frac{4b_4+2b_2+b_1+2+\varepsilon}{3} \leq \frac{4b_4+2b_2+2+2b_4+b_2+1}{3} \leq 2b_4+b_2+1$; thus, at worst, $2b_4+b_2+1$ is exactly "big enough," and we assign a preliminary set $W_{c'}$ the color c', such that (a) all vertices that are on a 4-thread incident with x or y at a distance of 3 from x or y, respectively, are in $W_{c'}$ and (b) some nonleaf vertices adjacent to x or y are in $W_{c'}$ such that $W_{c'}$ has size $p_{c'}$. (This is possible because again $b_4 < s/3$.)

The remaining vertices, which we shall group together in a set $W_{c''}$, are "assigned" the color c'', with the caveat that these $W_{c''}$ vertices contain the leaves and thus may not have c'' in their list.

To pass, therefore, to a legitimate L-coloring, we pair the vertices of $W_{c''}$ that are leaves with a subset of $W_{c'}$ as follows. For each z that is a leaf of a 2-thread or a 4-thread, define z^* to be the neighbor of z. For each z that is a leaf of a 1-thread, we may assign a unique z^* such that z^* is a neighbor of x or y and that z^* lies in a 4-thread or such that z^* is the neighbor of a leaf colored by c. (Note that this is possible because $b_4 + |\{\text{leafs colored } c\}| = p_c \geq p_{c''}$.) Now, in every case, $z \in W_{c''}$ and $z^* \in W_{c'}$ by construction; swap the colors on z and z^* if c'' is not in ℓ_z. Note that the obtained coloring is legitimate because in each case, the other vertices adjacent to z^* received the color c. ∎

5. Equitable 3-coloring.

In this section, we prove Theorem 1.3.

By Theorem 1.2, we need to show only that planar graphs with minimum degree at least 2 and girth at least 14 are equitably 3-colorable. Suppose not, and let G be a counterexample with $|V| + |E|$ as small as possible. The proof of the following claim is essentially a line-by-line copy of the proof of Claim 3.1, so we omit it.

CLAIM 5.1. *G has no t-thread where $t = 3$ or $t \geq 5$ and no thread with the same endvertices.*

Similarly to section 3, for a vertex x, let $T(x) = (a_4, a_2, a_1, a_0)$, where a_i is the number of i-threads incident to x, and let $t(x) = 4a_4 + 2a_2 + a_1$.

CLAIM 5.2. *Let x be a vertex with $3 \leq d(x) \leq 6$. Then*
(a) *if $d(x) = 3$, then either $t(x) \leq 4$ or $T(x) = (1, 0, 2, 0)$;*
(b) *if $d(x) = 4$, then $t(x) \leq 7$ or $T(x) = (2, 0, 0, 2)$;*
(c) *if $d(x) \in \{5, 6\}$, then $a_4 \leq d(x) - 2$.*

Proof. Assume, for a contradiction, that (i) $d(x) = 3$ and $t(x) \geq 5$ (ii) $d(x) = 4$ and $t(x) \geq 8$, or (iii) $d(x) \in \{5, 6\}$ and $a_4 \geq d(x) - 1$.

Note that if $d(x) \in \{3, 5, 6\}$, $a_0 \leq 1$. If $d(x) = 4$, then $a_0 > 1$ and $t(x) \geq 8$ only if $a_4 = a_0 = 2$, in which case $T(x) = (1, 0, 2, 0)$, as wanted. So we may assume that $a_0 \leq 1$, and thus Lemma 4.1 applies.

Let H be the subgraph of G induced by x and its loosely adjacent 2-vertices. Then $G - H$ has an equitable 3-coloring f, and we may assume that f cannot be extended to H. Thus, by Lemma 4.1, $2a_4 + a_2 \leq a_1 + \epsilon$, where $\epsilon = 3\lceil \frac{|V(H)|}{3} \rceil - |V(H)|$. Since $t(x) = 4a_4 + 2a_2 + a_1$,

$$t(x) = 2(2a_4 + a_2) + a_1 \leq 3a_1 + 2\epsilon. \qquad (3)$$

Let $d(x) = 3$. By (3), $a_1 \geq 1$. Then $(a_4, a_2) \in \{(1, 1), (2, 0), (1, 0), (0, 2)\}$. If $(a_4, a_2) = (1, 1)$, then $\epsilon = 1$ and $a_1 = 1$, a contradiction to (3). If $(a_4, a_2) = (2, 0)$, then $a_1 = 1$ and $\epsilon = 2$, a contradiction to (3) again. And if $(a_4, a_2) = (0, 2)$, then $a_1 = 1$ and $\epsilon = 0$, another contradiction to (3). So $(a_4, a_2) = (1, 0)$. It follows that $a_1 = 2$ or $a_1 = a_0 = 1$. If $a_1 = a_0 = 1$, then $\epsilon = 0$, a contradiction to (3). Therefore $a_1 = 2$ and $T(x) = (1, 0, 2, 0)$.

Let $d(x) = 4$. By (3), $a_1 \geq 2$. Then $(a_4, a_2) \in \{(1, 1), (2, 0)\}$. Consequently, $a_1 = 2$. If $(a_4, a_2) = (1, 1)$, then $\epsilon = 0$, a contradiction to (3). If $(a_4, a_2) = (2, 0)$, then $\varepsilon = 1$, again a contradiction to (3).

If $d(x) \in \{5, 6\}$, then $a_1 \leq 1$, which contradicts (3). ∎

We call a 3-vertex x *bad* if $T(x) = (1, 0, 2, 0)$.

CLAIM 5.3. *Let x be a bad 3-vertex. Let y be a 3-vertex that is loosely 1-adjacent to x. Then*
(a) *y is not incident to a t-thread where $t \geq 2$; hence $t(y) \leq 3$; and*
(b) *x is the only bad 3-vertex to which y is loosely 1-adjacent.*

Proof. (a) Suppose that y is incident with a $t-$thread where $t \geq 2$. Let H be the subgraph of G induced by x, y, and all 2-vertices loosely adjacent to x or y. We apply Lemma 4.2 to H, observing that $b_4 = a_4^y + 1$, $b_2 = a_2^y$, and $b_1 = a_1^y + 1$. We find that H is reducible if $2b_4 + b_2 \geq b_1 - 1 + \varepsilon$ or equivalently if $2a_4^y + a_2^y \geq a_1^y + \epsilon - 2$.

Now if $a_1^y = 1$, then $2a_4^y + a_2^y \geq 1 \geq \epsilon - 1 = a_1^y + \epsilon - 2$. Hence, since y is adjacent to a t-thread with $t \geq 2$, it must be that $a_1^y = 2$. Thus we may reduce H if $2a_4^y + a_2^y \geq \varepsilon$. This is true if $a_4^y > 0$. Thus we may assume that $a_2^y = 1$. But in this case, $|H| = 11$, $\varepsilon = 1$, and $a_2^y \geq \varepsilon$. Thus H is reducible.

(b) Suppose now that y is also loosely 1-adjacent to another bad 3-vertex z. Let H be the subgraph induced by x, y, z and all the 2-vertices loosely adjacent to x, y, or z. Let G' be $G - H$. Note that by the girth condition, x and z may be loosely adjacent to the same vertex w through the 4-threads, but in that case, w cannot be a 3-vertex since otherwise it violates Claim 5.2(a). So $\delta(G') \geq 2$, and thus G' is equitably 3-colorable. We need to extend this equitable 3-coloring to all of G.

We will 3-color H, and for $i \in \{1, 2, 3\}$, let U_i be the set of vertices of H colored by i. For the coloring to remain equitable, we need $|U_1| \geq |U_2| \geq |U_3| \geq |U_1| - 1$. Call a proper coloring of H "good" if it satisfies $|U_1| \geq |U_2| \geq |U_3| \geq |U_1| - 1$.

The union of x, y, z together with the 1-threads at x and z forms a 9-path; let us label it as $v_1w_1xw_2yw_3zw_4v_2$. Label the 4-thread at x as $xx_1x_2x_3x_4v_3$, and label the 4-thread at z as $zz_1z_2z_3z_4v_4$.

First suppose that y is adjacent to a 0-thread. Then $|U_i|$ should be 5 for all i, and some color is disallowed at y by its adjacency in G to a vertex of G'. Assume, without loss of generality, that 3 is an allowed color at y. Let $U_1' = \{w_1, w_4, x_1, x_3, z_3\}$, $U_2' = \{w_2, w_3, x_4, z_1, z_4\}$, and $U_3' = \{x, y, z, x_2, z_2\}$. This is a good coloring of H, so it remains only to repair any conflicts at the leaves of H when H is attached to G'. Notice that if there is a conflict with the leaf adjacent to w_1, we may simply swap the colors on w_1 and w_2. Likewise, we may pair w_3 with w_4, x_3 with x_4, and z_3 with z_4, swapping any pair if there is a conflict at the associated leaf. Any such swap results in another good coloring of H, and swapping any pair does not interfere with any other pair. Thus we may obtain appropriate U_i in this case.

If y is incident to a third 1-thread with 2-vertex y_1, then we keep the U_i's as before and color y_1 by 1. Note that y_1 and z_1 form another swappable pair if there is a conflict at y_1.

By (a), y is not incident to any t-thread with $t \geq 2$, so the proof is complete. ∎

Since $g(G) \geq 14$, we have $mad(G) < \frac{7}{3}$. Let $M(x) = d(x) - \frac{7}{3}$ be the *initial charge* of x for $x \in V$. We will redistribute the charges among vertices according to the *discharging rules* below:

(R1) Every 3^+-vertex sends $\frac{1}{6}$ to each loosely adjacent 2-vertex.

(R2) Every 3^+-vertex sends $\frac{1}{6}$ to each loosely 1-adjacent bad 3-vertex.

Let $M'(x)$ be the final charge of x. The following claim shows a contradiction to (1), which in turn implies the truth of Theorem 1.3.

CLAIM 5.4. *For each* $x \in V$, $M'(x) \geq 0$.

Proof. If $d(x) = 2$, then $M'(x) = 2 - \frac{7}{3} + 2 \cdot \frac{1}{6} = 0$.

If $d(x) = 3$, then if x is bad, it gains $\frac{1}{6}$ from each of the two loosely 1-adjacent vertices, and thus $M'(x) \geq 3 - \frac{7}{3} - 6 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} = 0$. If x is not bad and is not loosely 1-adjacent to a bad vertex, then $M'(x) \geq 3 - \frac{7}{3} - 4 \cdot \frac{1}{6} = 0$ by Claim 5.2. If x is not bad and is loosely 1-adjacent to a bad 3-vertex, then $t(x) \leq 3$ by Claim 5.3, and thus $M'(x) \geq 3 - \frac{7}{3} - 3 \cdot \frac{1}{6} - \frac{1}{6} = 0$.

For $d(x) \geq 4$, note that $M'(x) \geq d(x) - \frac{7}{3} - \frac{(4a_4+2a_2+2a_1)}{6}$. Since $d(x) = a_4 + a_2 + a_1 + a_0$,

$$M'(x) \geq \frac{1}{3}(2d(x) - 7 - a_4 + a_0)\,.$$

If $d(x) \geq 7$, then $M'(x) \geq (d(x) - a_4 + a_0)/3 \geq 0$. If $d(x) \in \{5, 6\}$,then Claim 5.2 implies that $a_4 \leq d(x) - 2$, and thus $M'(x) \geq 0$.

Assume now that x has degree 4. To show that $M'(x) \geq 0$, it suffices to show that $a_4 \leq a_0 + 1$, which is true since Claim 5.2 ensures that $a_4 \leq 1$, or $(a_4, a_0) = (2, 2)$. ∎

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